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Supersymmetry in singular spaces

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Abstract: We develop *the concept of supersymmetry in singular spaces*, apply it in an example for 3-branes in $D = 5$ and comment on 8-branes in $D = 10$. The new construction has an interpretation that the brane is a sink for the flux and requires adding to the standard supergravity a $(D-1)$-form field and a supersymmetry singlet field. This allows a consistent definition of supersymmetry on a $S_1/\mathbb{Z}_2$ orbifold, the bulk and the brane actions being separately supersymmetric. Randall-Sundrum brane-worlds can be reproduced in this framework without fine tuning. For fixed scalars, the doubling of unbroken supersymmetries takes place and the negative tension brane can be pushed to infinity. In more general BPS domain walls with 1/2 of unbroken supersymmetries, the distance between branes in some cases may be restricted by the collapsing cycles of the Calabi-Yau manifold. The energy of any static x^5-dependent bosonic configuration vanishes, $E = 0$, in analogy with the vanishing of the hamiltonian in a closed universe.

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Contents

1. Introduction

Usually, supersymmetry is associated with non-negative energy. In asymptotically flat spaces this is related to the time translation operator which is a square of fermionic operators $E = Q^2 \geq 0$. In curved space the positivity of energy theorem is proved via Nestor-Israel-Witten construction where it is usually assumed that the space is non-singular. In case of black holes with the singularity covered by the horizon, the argument about the positivity of energy was also extended [1].

Recently there were number of reasons to reconsider the issues of supersymmetry, in general. In the brane world scenarios with fine tuning where singular branes are introduced [2, 3], the status of the supersymmetric embedding was not clear. The first attempts to relax the fine tuning was to find smooth solutions of supergravity where domain walls are build from some scalar fields [4]–[8]. With respect to the 'alternative to compactification scenario' [3], no-go theorems were established [9, 10] for the BPS smooth solutions of a certain class of supergravity theories ($N = 2$ with vector [11, 12] and tensor multiplets [13]) on the basis of the negative definiteness of the derivative of the β-function in the relevant renormalization group behavior (UV behavior).

More recently, the complete $N = 2$ supergravity theory was constructed in [14] where also the hypermultiplets are included. The status of hypermultiplets in the adS brane world is not yet settled,[1] more work is required. Even if the solutions with the IR fixed point will be found, there will still be a problem to find a configuration in a smooth supergravity which will connect two such IR fixed points. This may be also explained following a suggestive argument:[2] BPS solutions are expected to be given by harmonic functions, in codimension 1 they must have a kink. A kink can not appear as a smooth solution. The complimentary argument was given in [15]: the fermion mass does not change sign when passing through the wall, which may explain the absence of smooth solutions with the required properties. The version of a no-go theorem for smooth brane worlds from compactifications, under some assumptions about the potential, was recently proposed in [16]. None of these arguments seems to give an unconditional final proof that there are no smooth domain walls for a brane world scenario. However, they suggest to find out whether the clear framework is available for the non-smooth supersymmetric solutions.

Thus the purpose of this paper is to generalize supersymmetry for singular spaces, where the curvature may have some δ-function singularities. In singular spaces there was no consistent and complete definition of supersymmetry so far[3] and the related issue of non-negative energy was not clearly addressed.

An important *reason to reformulate supersymmetry in singular spaces is related to supersymmetric domain walls*: objects of codimension $d = 1$, i.e. $(D-2)$-branes in D-dimensional space. They may be associated with D-form fluxes which are dual to a scalar and piecewise constant. Such forms do not fall off with distance. In an infinite volume, they may lead to an infinite energy and would be unphysical. Therefore, objects like the 8-brane in $D = 10$ and 3-branes in $D = 5$ may not exist

[1] J. Louis, talk at SUSY2K, July 2000.

[2] K. Stelle, talks at Fradkin and Gürsey conferences, June 2000.

[3] We will comment on the available information [17]–[22] below.

as independent objects. They may need some planes that serve as sinks for the fluxes for supersymmetric configurations of codimension 1 [23]. Note that the fluxes in the branes of higher codimension do vanish at infinity and this problem can be avoided.

Usually local supersymmetry is realized in supergravity when the lagrangian is integrated over a continuous space. Under supersymmetry, the lagrangian transforms as a total derivative. The parameters of local supersymmetry are assumed to fall off at infinity and therefore the action is invariant. If in addition to the supergravity action one considers the κ-symmetric worldvolume action of the *positive tension brane* of codimension $d \geq 3$, it provides the δ-functional sources for the harmonic functions that describe the configuration, $\partial_i \partial_i H = T \delta^d(\vec{y})$.

In the case of branes of codimension $d = 1$ defined on an orbifold $S^1/\mathbb{Z}_2$, the harmonic functions satisfy the equations $\partial_y \partial_y H = T\delta(y) - T\delta(y - \tilde{y})$. This can be seen e.g. in the Hořava-Witten (HW) construction [17] developed for 5-dimensional 3-branes in [18]. The metric of such solutions depends on $H = c + T|y|$ and has two kinks at the orbifold fixed points: one at $y = 0$ and the other at $y = \tilde{y}$ where $\tilde{y}$ is identified with $-\tilde{y}$. The supersymmetry in HW construction includes the contribution from anomalies and requires quantum consistency. It has some problematic features related to higher order corrections and quartic fermionic terms.

To set up a new general point of view on supersymmetry in singular spaces, and to clarify the issue of energy of brane worlds, we introduce here a set of rules defining consistent supersymmetry on singular spaces.

The most important new steps include: (i) some constants (masses or gauge couplings) have to become "$\mathbb{Z}_2$-odd" to make supersymmetry commuting with $\mathbb{Z}_2$-symmetry. This can be achieved by promoting such a constant to the status of a supersymmetry singlet field, as suggested in [24]; (ii) moreover, one has to add to the theory the $(D-1)$-form potential[4] to compensate the variation of the lagrangian proportional to the derivative of the new field; (iii) one can find afterwards the new bulk and the brane actions, which are separately invariant under supersymmetry.

The *unusual features of the new supersymmetry* are the presence of the supersymmetry singlet field and $(D-1)$-form field in the bulk, and the fact that the purely bosonic action on the brane is supersymmetric due to the fact that its fermionic partner is a $\mathbb{Z}_2$-odd fermion which vanishes on the brane.

In absence of brane actions, the new fields become irrelevant: on shell for the 4-form and supersymmetry singlet, the bulk action reduces to the standard supergravity action supersymmetric under the standard rules.

[4]The importance of the $(D-1)$-form field was realized by Duff and van Nieuwenhuizen [25], who pointed out 20 years ago the quantum inequivalence of the theories with and without such field in the context of trace anomalies and 1-loop counterterms in topologically non-trivial backgrounds. At about the same time Aurilia, Nicolai and Townsend [26] have found that the $(D-1)$-form, which propagates no physical particles, carries a surprising physics. They have looked at θ parameter in QCD and at the cosmological constant in supergravity.

In general, when the brane actions are added, the new fields play an important role in understanding the energy issue. We will find that *the total energy of supersymmetric configurations vanishes locally at each brane. The positive (negative) energy of the brane tensions are compensated separately by the terms with the derivative of the supersymmetry singlet field.* The energy of any static y-dependent bosonic configuration vanishes locally, $E = 0$, in analogy with the vanishing of the hamiltonian in a closed universe.[5]

The strategy is applied in detail in the particular case of a 3-brane in $D = 5$ on the basis of U(1) gauged $N = 2$ supergravity interacting with abelian vector multiplets. We expect that it will work in other cases as well: in $D = 5$ one can try to include more general gauging and tensor and hypermultiplets. The gauged $N = 4$ and $N = 8$ supergravities in $D = 5$ are also natural candidates for an analogous extension. We will give a brief discussion of the particularly interesting case of the 8-brane in $D = 10$.

The paper has two main parts.

In section 2, **The supersymmetric theory: bosons and fermions**, we construct the supersymmetric actions in the bulk and in the brane. To do so, we first identify the $\mathbb{Z}_2$ operator in section 2.1. Then, we construct the supersymmetric theory with three steps in section 2.2. The final result is written down in section 2.3

In section 3, **The background: vanishing fermions, bosons solve equation of motion**, we study bosonic solutions of the theory from section 2 and their unbroken supersymmetries. Section 3.1 discusses the vanishing of the energy. The BPS equations and preserved supersymmetries in these singular spaces are discussed in section 3.2. We consider two cases, the fixed scalars, with doubling of supersymmetry, and the general stabilization equations in very special geometry, with 1/2 preserved supersymmetry. We show the resulting formulas for the example of the STU wall, and for a particular Calabi-Yau wall in section 3.3.

Finally, in section 4, we discuss the similar mechanism for the 8-brane in 10 dimensions.

The notations and use of indices are presented in appendix A. For those unfamiliar with the 5-dimensional supergravity, and for establishing the related notations, we present its structure in appendix B. In appendix C we discuss the previous attempts to define supersymmetry on orbifolds.

2. The supersymmetric theory: bosons and fermions

2.1 Local supersymmetry and $\mathbb{Z}_2$

Our setup has the following basic features.

[5]We are grateful to A. Linde who suggested this analogy.

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1. We start with the standard action S_{bulk} of five-dimensional gauged supergravity with a symplectic Majorana supersymmetry with parameters ϵ^i, $i = 1, 2$, thus having 8 real components, coupled with an arbitrary number of vector multiplets [11, 12].

2. *Orbifold construction.* Fields live on a circle in the direction x^5, with an orbifold condition. The circle implies that $\Phi(x^5) = \Phi(x^5 + 2\tilde{x}^5)$, where $\tilde{x}^5$ is some arbitrary parameter setting the length of the circle. We use a concept of parity to split the fields in even and odd under a $\mathbb{Z}_2$:

$$\Phi_{\text{even}}(-x^5) = \Phi_{\text{even}}(x^5)\,, \qquad \Phi_{\text{odd}}(-x^5) = -\Phi_{\text{odd}}(x^5)\,. \tag{2.1}$$

   This implies that the odd fields vanish at $x^5 = 0$ and at $x^5 = \tilde{x}^5$, where we will put the branes. The supersymmetries are also split in half even (ϵ_+) and half are odd (ϵ_-). Both $\epsilon_\pm$ have 4 real components. The bulk action is even, and all transformation rules are consistent with the assignments.

3. *The brane action* S_{brane} is introduced. We place two branes, one at $x^5 = 0$ and another one at $x_5 = \tilde{x}_5$. The actions depend on the values of the bulk fields at $x^5 = 0$ and $x_5 = \tilde{x}_5$. As the odd fields are zero on the brane, the brane action only depends on the even fields. Only the supersymmetries ϵ_+ act on these fields.

We analyse all possibilities to make parity assignments consistent with supersymmetry commuting with $\mathbb{Z}_2$-symmetry. We conclude that the consistent supersymmetry on orbifolds without brane actions does not fix the $\mathbb{Z}_2$-properties of the gauge coupling. However, after adding the brane action we must require the gauge coupling to be $\mathbb{Z}_2$-odd.[6]

We have to treat the fact that g is *only piecewise constant.* This approach is inspired by [24]. It consists of the following steps

1. Replace in the action the constant g by a scalar function $G(x)$ and keep this field a *supersymmetry singlet.* The supersymmetry singlet field was introduced in the context of κ-symmetric brane actions in [27]. This field has some peculiar properties: its shift under translation vanishes on shell but not off shell.

2. Add a Lagrange multiplier $A_{\mu\nu\rho\sigma}$ that imposes the constraint $\partial_\mu G = 0$. At this point this reproduces the known 'bulk' actions of 5 dimensions [11, 12, 13, 14]. The G-field dependent action is invariant by appropriately choosing supersymmetry transformations as well as local U(1) gauge transformations for $A_{\mu\nu\rho\sigma}$.

[6]The second possibility, $\mathbb{Z}_2$-even gauge coupling, as chosen in [19], will be discussed in appendix C.

3. Add source terms that are separately invariant, but may contain $\delta(x^5)$ and $\delta(x^5-\tilde{x}^5)$ and are dependent on $A_{\underline{\mu\nu\rho\sigma}}$, where these are the components in the 4 dimensions of the brane. After these additions the $A_{\underline{\mu\nu\rho\sigma}}$ field equation says that $\partial_5 G(x) \propto \delta(x^5)-\delta(x^5-\tilde{x}^5)$, and thus $G(x)\propto \varepsilon(x^5)$. This provides a supersymmetric mechanism to change the sign of the coupling constant (and of the fermion mass) when passing through the wall.

We will first consider the possibilities for a $\mathbb{Z}_2$ in the 5-dimensional action. Define the operator Π for any field, being $+1$ or -1 whether it is even or odd under $x^5\to -x^5$. For the symplectic Majorana spinors, the splitting might involve some projection operators. Let us look for a $\mathbb{Z}_2$ of the form (for any symplectic Majorana spinor λ^i)

$$\lambda^i(x^5)=\Pi(\lambda)\gamma_5 M^i{}_j\lambda^j(-x^5)\,, \tag{2.2}$$

where $\Pi(\lambda)$ is thus a number $+1$ or -1, while $M^i{}_j$ is so far an undetermined 2×2 matrix. If this has to be a $\mathbb{Z}_2$, the operation should square to $\mathbb{1}$. Therefore M should square to $\mathbb{1}$. Notice that this is independent of whether we included γ_5 in (2.2) or not. Next, this has to be consistent with the reality condition (symplectic Majorana condition, see appendix A). This implies that M should satisfy $M^C\equiv\sigma_2 M^*\sigma_2 = -M$, or thus, $M=\mathrm{i}m_0\mathbb{1}+m_1\sigma_1+m_2\sigma_2+m_3\sigma_3$ with m_0, m_1, m_2 and m_3 real. The $M^2=\mathbb{1}$ condition implies that

$$M^i{}_j=m_1(\sigma_1)^i{}_j+m_2(\sigma_2)^i{}_j+m_3(\sigma_3)^i{}_j\,,\qquad \text{with } m_1,\,m_2,\,m_3\in\mathbb{R}\,, \tag{2.3}$$

which means that M is hermitean and traceless. If γ_5 were not included in (2.2), the numbers m would have to be pure imaginary, and the $M^2=\mathbb{1}$ condition would imply $M=\mathbb{1}$, but then we would have no projection at all. Thus we conclude that (2.2) with (2.3) is the only one that is possible.

Lowering the two indices on M, it becomes a symmetric matrix:

$$M_{ij}=-m_1(\sigma_3)_{ij}-\mathrm{i}m_2(\mathbb{1})_{ij}+m_3(\sigma_1)_{ij}\,. \tag{2.4}$$

The parity will be related to the fifth direction. We will therefore assign a negative parity to the x^5 coordinate, or $\Pi(\partial_5)=-1$. Consider now the supersymmetry transformation laws in the bulk, see (B.12). For one of the fermions we may arbitrary assign even parity, e.g. for the components of the gravitino in the directions excluding '5', i.e. $\psi_{\underline{\mu}}$. Consider first the supersymmetry transformation laws that are independent of the gauge coupling g. The consistency of the parity assignments determines

$$\begin{aligned} &\Pi(e_{\underline{\mu}}{}^m)=1\,, && \Pi(e_5{}^5)=1\,, && \Pi(A^I_5)=1\,,\\ &\Pi(e_5{}^m)=-1\,, && \Pi(e_{\underline{\mu}}{}^5)=-1\,, && \Pi(A^I_{\underline{\mu}})=\Pi(\Lambda^I)=-1\,,\\ &\Pi(\psi_{\underline{\mu}})=\Pi(\epsilon)=1\,, && \Pi(\psi_5)=\Pi(\lambda)=-1\,. \end{aligned} \tag{2.5}$$

Note that also the supersymmetry parameters got a parity projection and that the parameters of the gauge transformations, Λ^I, have to be odd.

Now we consider the terms with g in the gravitino transformation law (B.12). They depend on the constant matrix Q_{ij}, see (B.6). Taking the parity transformation of both sides, we find that they are proportional to

$$M^j{}_i Q^k{}_j \epsilon_k = -\Pi(g) Q^j{}_i M^k{}_j \epsilon_k \,, \tag{2.6}$$

where we allowed a parity transformation of g. We find that if Q and M commute (which means that they are proportional) one needs $\Pi(g) = -1$, while if they anticommute (thus are taken in orthogonal directions in the SU(2)-space), $\Pi(g) = 1$.

Taking an anticommuting M and Q brings us to the setup of [19]. We will see below, that the addition of brane actions forces us to take a matrix that commutes with Q. If Q and M commute, one has two possibilities to implement the parity assignment $\Pi(g) = -1$. In the approach of [22], who take $M^i{}_j = (\sigma_3)^i{}_j$, this assignment is realized by replacing g by $g\varepsilon(x^5)$. On the other hand, we will be able to make this assignment by promoting g to a field.

To continue in the direction of [19, 20, 22], one has to provide the consistent definition of supersymmetry with step functions and delta functions present in supersymmetry rules and in the action. The construction of the higher order fermionic terms and the structure of the algebra of supersymmetry in singular spaces in these approaches are difficult as the HW theory shows. Instead of this, a new way to introduce supersymmetry in singular spaces will be developed below.

2.2 Supersymmetry in a singular space

2.2.1 Step 1: the bulk action

We consider 5-dimensional supergravity coupled to n vector multiplets. The coupling is determined by a 3-index real symmetric constant tensor C_{IJK}, where the indices run over $n+1$ values. We consider a U(1) gauging of the R-symmetry group determined by real constants V_I. The direction in SU(2) space is determined by a matrix $Q_i{}^j$. Obviously, the choice of that direction has no physical consequences. The full action, to which we will refer as the GST action [12], and transformation rules, are given in appendix B. Replacing the coupling constant by a scalar $G(x)$, the action is not any more invariant supersymmetry,

$$e^{-1}\delta(\epsilon)\mathcal{L}_{GST} = \left[-\mathrm{i}\frac{3}{2}\overline{\psi}^i_\mu\gamma^{\mu\nu}\epsilon^j W - \overline{\psi}^i_\mu\gamma^{\mu\nu\rho}\epsilon^j A^{(R)}_\rho + \frac{3}{2}\overline{\lambda}^i_x W^{,x}\gamma^\nu\epsilon^j\right] Q_{ij}\partial_\nu G \,, \tag{2.7}$$

and neither under the U(1) gauged symmetry,

$$e^{-1}\delta_R\mathcal{L}_{GST} = \frac{1}{2}\left[\bar{\psi}^i_\mu\gamma^{\mu\nu\rho}\psi^j_\rho + \overline{\lambda}^i_x\gamma^\nu\lambda^{jx}\right] Q_{ij}\Lambda_R\partial_\nu G \,, \tag{2.8}$$

where Λ_R is the parameter of the R-symmetry

$$\Lambda_R = V_I\Lambda^I \,. \tag{2.9}$$

2.2.2 Step 2: the four-form

In the second step, we add the following Lagrange multiplier term:

$$S_A = \frac{1}{4!}\int \mathrm{d}^5x\,\varepsilon^{\mu\nu\rho\sigma\tau}A_{\mu\nu\rho\sigma}\partial_\tau G\,. \tag{2.10}$$

Now we can make the action invariant under supersymmetry. This is obtained 1) by taking G invariant under supersymmetry, and 2) by defining the variation of A such that all $\partial_\mu G$ terms in the transformation of the rest of the action are cancelled. The fact that G is invariant is consistent with the algebra because the translation of G is a field equation (G is a supersymmetry singlet [27]). Thus the algebra is realized on-shell. The resulting variation of $A_{\mu\nu\rho\sigma}$ under supersymmetry is

$$\delta(\epsilon)A_{\mu\nu\rho\sigma} = \left[-6\bar{\epsilon}^i\gamma_{[\mu\nu\rho}\psi^j_{\sigma]}W + \mathrm{i}\bar{\epsilon}^i\gamma_{[\mu\nu}A^{(R)}_\rho\psi^j_{\sigma]} - \mathrm{i}\frac{3}{2}\bar{\epsilon}^i\gamma_{\mu\nu\rho\sigma}\lambda^j_x W^{,x}\right]Q_{ij}\,. \tag{2.11}$$

Under gauge transformations the 4-form transforms as follows

$$\frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma\tau}\delta_R A_{\mu\nu\rho\sigma} = -\frac{1}{2}e\left[\bar{\psi}^i_\mu\gamma^{\mu\tau\rho}\psi^j_\rho + \overline{\lambda}^i_x\gamma^\tau\lambda^{jx}\right]Q_{ij}\Lambda_R\,. \tag{2.12}$$

We define the covariant flux $\hat{F}$ as follows:

$$\begin{aligned}\hat{F} \equiv{}& \frac{1}{4!}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\partial_\mu A_{\nu\rho\sigma\tau} + \frac{1}{2}\bar{\psi}^i_\mu\gamma^{\mu\nu\rho}A^{(R)}_\nu Q_{ij}\psi^j_\rho + \frac{1}{2}\overline{\lambda}^i_x\gamma^\mu A^{(R)}_\mu Q_{ij}\lambda^{jx} + \\ &+\frac{3}{2}\left[\overline{\lambda}^i_x\gamma^\mu\psi^j_\mu W^{,x} - \mathrm{i}\frac{1}{2}\overline{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu W\right]Q_{ij}\,.\end{aligned} \tag{2.13}$$

The closure of the algebra on shell is due to a G field equation

$$\hat{F} = 12G\left(W^2 - \frac{3}{4}\left(\frac{\partial W}{\partial\varphi^x}\right)^2\right) + \mathrm{i}\overline{\lambda}^{ix}\lambda^{jy}\left(-\frac{1}{4}g_{xy}W + \sqrt{\frac{3}{2}}T_{xyz}W^{,z}\right)Q_{ij}\,, \tag{2.14}$$

where the bosonic part is related to the potential, given by

$$V = -6G^2\left[W^2 - \frac{3}{4}\left(\frac{\partial W}{\partial\varphi^x}\right)^2\right]. \tag{2.15}$$

This equation relates the flux to the potential via the singlet field G (up to terms with fermions). We will see later that on shell the flux will change sign when passing through the wall. This will explain the role of the wall as a sink for the flux.

Remarks:

- the action is invariant under 8 supersymmetries.

- the lagrangian is invariant up to a total derivative. This is sufficient if the fields either drop off at infinity (as it is supposed to be in the 4-dimensional spacetime), or the space is cyclic and the fields are continuous (as it is supposed to be in the x^5 direction). To have at the end $G(x)$ piecewise constant, but not everywhere the same, it is clear that we need at least 2 branes where it can jump.

- The procedure outlined in step 2 is very general and does not depend on the details of the configuration.

We have explained in section 2.1 that the GST-action allows two different parity assignments for G. The action (2.10) respects both choices with the assignments

$$\begin{cases} \Pi(G) = -1\,, & \Pi(A_{\underline{\mu\nu\rho\sigma}}) = +1\,, \qquad \Pi(A_{\underline{\mu\nu\rho}5}) = -1\,, \\ \Pi(G) = +1\,, & \Pi(A_{\underline{\mu\nu\rho\sigma}}) = -1\,, \qquad \Pi(A_{\underline{\mu\nu\rho}5}) = +1\,. \end{cases} \tag{2.16}$$

2.2.3 Step 3: the brane as a sink for the flux

If we have only the bulk action, the field equation still implies that G is constant everywhere. The flux is proportional to the potential and on shell for $A_{\mu\nu\rho\sigma\tau}$ and G we recover the standard supergravity.

We thus *need sources to modify the field equation on G*. These we can choose according to a physical situation. For reproducing the scenario described above, we take two branes positioned at $x_5 = 0$ and at $x_5 = \tilde{x}_5$. For both branes we introduce a worldvolume action, which basically is the Dirac-Born-Infeld action in a curved background with all excitations of the worldvolume fields set equal to zero.

The brane action includes the pullback of the metric, the scalars and the 4-form of the bulk action. The coefficient of the determinant of the metric is taken to be the function W. It is related to the central charge, similar to what was obtained for black holes in $N = 2$, $d = 4$ in [28]. The Wess-Zumino term describes the charge of the domain wall.

Remember that, as explained after (2.1), odd fields vanish on the branes. Therefore, if we want to use the pullback of the components $A_{\underline{\mu\nu\rho\sigma}}$ on the brane, we need that their parity is even. This forces us to take the first choice in (2.16). This is consistent with the scenario of [22], but it is problematic to incorporate the approach of [19] in our framework of 'consistent supersymmetry on singular spaces'.

The brane action is

$$S_{\text{brane}} = -2g \int d^5x \, \left(\delta(x^5) - \delta(x^5 - \tilde{x}^5)\right) \left(e_{(4)} 3\alpha W + \frac{1}{4!}\varepsilon^{\underline{\mu\nu\rho\sigma}} A_{\underline{\mu\nu\rho\sigma}}\right), \tag{2.17}$$

where $e_{(4)}$ is the determinant of the 4 by 4 vierbein $e_{\underline{\mu}}^{m}$, and $\varepsilon^{\underline{\mu\nu\rho\sigma}} = \varepsilon^{\underline{\mu\nu\rho\sigma}5}$ is in the same way a 4-density. The factor $\alpha = \pm 1$ is a sign to be chosen later. The new field equation for $A_{\underline{\mu\nu\rho\sigma}}$ is

$$\partial_5 G(x^5) = 2g \left(\delta(x^5) - \delta(x^5 - \tilde{x}^5)\right), \tag{2.18}$$

which has as solution

$$G(x) = g\varepsilon(x^5)\,, \tag{2.19}$$

for $-\tilde{x}^5 < x^5 < \tilde{x}^5$. One should understand $\varepsilon(x^5)$ as the function that is $+1$ for $0 < x^5 < \tilde{x}^5$, and -1 for $-\tilde{x}^5 < x^5 < 0$. Thus it has also a jump at $x^5 = \tilde{x}^5 \equiv -\tilde{x}^5$, and

$$\frac{1}{2}\partial_5 \varepsilon(x^5) = \delta(x^5) - \delta(x^5 - \tilde{x}^5)\,. \tag{2.20}$$

Now we may look at the equations of motion for the flux with account of the value of the on-shell G-field. The bosonic part of the flux (2.14) is on-shell

$$\hat{F}_{\text{on-shell,bos}} = \varepsilon(x^5) 12g\left[W^2 - \frac{3}{4}\left(\frac{\partial W}{\partial \varphi^x}\right)^2\right]. \tag{2.21}$$

Clearly the flux is changing the sign when passing through the brane, which justifies the title of this section. This may be contrasted with the properties of the potential when the G-field is on shell:

$$V_{\text{on-shell}} = -6g^2\left[W^2 - \frac{3}{4}\left(\frac{\partial W}{\partial \varphi^x}\right)^2\right], \tag{2.22}$$

where the standard assumption is made that $\varepsilon^2(x^5) = 1$. The potential does not care about the existence of the brane.

The fermion mass terms on shell for the G-field also change the sign across the wall, as it follows from the terms in the action that are quadratic in fermions and linear in G.

We now consider the invariance of the brane action. The fünfbein satisfies $e_5^m = e_{\underline{\mu}}^5 = 0$. This is due to the parity assignment and orbifold condition, which implies that only even parity fields are non-zero on the brane. The variation of the brane action is ($A_{\underline{\mu}}^{(R)}$ is zero on the brane, and therefore those contributions can be neglected)

$$\begin{aligned}\delta S_{\text{brane}} = -3g \int \mathrm{d}^5 x \left(\delta(x^5) - \delta(x^5 - \tilde{x}^5)\right) \times \qquad\qquad \\ \times\, e_{(4)} \left[W \bar{\epsilon}^i \gamma^m e_m^{\underline{\mu}} \left(\alpha \psi_{\underline{\mu} i} - \mathrm{i}\gamma_5 Q_{ij} \psi_{\underline{\mu}}^j\right) + W_{,x} \bar{\epsilon}^i \left(\mathrm{i}\alpha \lambda_i^x - \gamma_5 Q_{ij} \lambda^{xj}\right)\right].\end{aligned} \tag{2.23}$$

This vanishes if we apply the $\mathbb{Z}_2$ projections of section 2.1 with

$$M_{ij} = \mathrm{i}\alpha Q_{ij}\,, \qquad \alpha = \pm 1\,. \tag{2.24}$$

It thus implies that

$$\begin{aligned}\psi_{\underline{\mu} i}(x^5) &= \mathrm{i}\alpha Q_{ij} \gamma_5 \psi_{\underline{\mu}}^j(-x^5)\,,\\ \lambda_i(x^5) &= -\mathrm{i}\alpha Q_{ij} \gamma_5 \lambda^j(-x^5)\,,\\ \epsilon_i(x^5) &= \mathrm{i}\alpha Q_{ij} \gamma_5 \epsilon^j(-x^5)\,,\end{aligned} \tag{2.25}$$

such that (2.23) vanishes.

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2.3 Summary of $d = 5$ supersymmetry on $S^1/\mathbb{Z}_2$

In summary, the new lagrangian in a singular space with the new supersymmetry is given by

$$S_{\text{new}}(x^5) = S_{\text{bulk}} + S_{\text{brane}}\,, \qquad S_{\text{bulk}} = S_{GST}(g \to G(x)) - \int \mathrm{d}^5x\, e\, F(x)G(x)\,. \quad (2.26)$$

Here F is a curl of the 4-form $F = \frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma\tau}\partial_\tau A_{\mu\nu\rho\sigma}$ and $S_{GST}(g) = S_0 + gS_1 + g^2S_2$ is the standard U(1) gauged supergravity of [12], see appendix B. The standard supergravity at $g = 0$ is called ungauged supergravity. We will use the notation $\mathcal{L}_0$ for its lagrangian. Our new bulk theory has lagrangian $\mathcal{L}_0$, there are terms linear in $G(x)$, which are proportional to the flux $\hat{F}$ defined in (2.13), and terms quadratic in the field $G(x)$, see (2.15),

$$\begin{aligned}\mathcal{L}_{\text{bulk}} &= \mathcal{L}_0 + 6eG^2(x)\left(W^2 - \frac{3}{4}W_{,x}^2\right) - e\,G(x)\hat{F} + \\ &\quad + e\mathrm{i}G(x)Q_{ij}\overline{\lambda}^{ix}\lambda^{jy}\left(-\frac{1}{4}g_{xy}W + \sqrt{\frac{3}{2}}T_{xyz}W^{,z}\right). \end{aligned} \quad (2.27)$$

In addition, we will denote by δ_0 the part of supersymmetry that acts at $g = 0$ and which forms the supersymmetry transformations of ungauged supergravity. For completeness we repeat here the brane actions.

$$\mathcal{L}_{\text{brane}} = -2g\left(\delta(x^5) - \delta(x^5 - \tilde{x}^5)\right)\left(e_{(4)}3W + \frac{1}{4!}\varepsilon^{\underline{\mu\nu\rho\sigma}}A_{\underline{\mu\nu\rho\sigma}}\right), \quad (2.28)$$

The new supersymmetry rules are

$$\begin{aligned}\delta(\epsilon)\{e_\mu^m, A_\mu^I, \varphi^x\} &= \delta_0\{e_\mu^m, A_\mu^I, \varphi^x\}\,,\\ \delta(\epsilon)\psi_{\mu i} &= \delta_0(\epsilon)\psi_{\mu i} + G(x)A_\mu^{(R)}Q_{ij}\epsilon^j + \mathrm{i}\frac{1}{2}G(x)\,\gamma_\mu\epsilon^j W Q_{ij}\,,\\ \delta(\epsilon)\lambda_i^x &= \delta_0(\epsilon)\lambda_i^x - \frac{3}{2}G(x)\epsilon^j W^{,x}Q_{ij}\,,\\ \frac{1}{4!}\delta(\epsilon)A_{\mu\nu\rho\sigma} &= \left[-6\overline{\epsilon}^i\gamma_{[\mu\nu\rho}\psi_{\sigma]}^j W + \mathrm{i}\overline{\epsilon}^i\gamma_{[\mu\nu}A_\rho^{(R)}\psi_{\sigma]}^j - \mathrm{i}\frac{3}{2}\overline{\epsilon}^i\gamma_{\mu\nu\rho\sigma}\lambda_x^j W^{,x}\right]Q_{ij}\\ \delta(\epsilon)G &= 0\,.\end{aligned} \quad (2.29)$$

The new gauge R-symmetry transformations are

$$\begin{aligned}\delta_R A_\mu^{(R)} &= \partial_\mu\Lambda_R\,,\\ \delta_R\psi_\mu^i &= G(x)\Lambda_R Q^{ij}\psi_{\mu j}\,,\\ \delta_R\lambda^{xi} &= G(x)\Lambda_R Q^{ij}\lambda_{\mu j}^x\,,\\ \varepsilon^{\mu\nu\rho\sigma\tau}\delta_R A_{\mu\nu\rho\sigma} &= -\frac{1}{2}\bar{\psi}_\mu^i\gamma^{\mu\tau\rho}\Lambda_R Q_{ij}\psi_\rho^j - \frac{1}{2}\overline{\lambda}_x^i\gamma^\tau\Lambda_R Q_{ij}\lambda^{jx}\,.\end{aligned} \quad (2.30)$$

The action is defined by an integral over a product space $\mathbf{M}$ of the 4-dimensional manifold and an orbifold, $\mathbf{M} = \mathbf{M}_4 \times S^1/\mathbb{Z}_2$.

$$S^{\rm new} = \int_{\mathbf{M}} d^4x dx^5 \mathcal{L}^{\rm new} \,. \tag{2.31}$$

The 4d manifold is non-compact and the 5th dimension is compact but has no boundaries in $S^1/\mathbb{Z}_2$. Therefore, all surface terms in the variation of the action vanish assuming as usual that the parameters of supersymmetry decrease at infinities of the 4d space. The bulk and the brane actions are separately invariant. The supersymmetry transformations form an on-shell closed algebra.

3. The background: vanishing fermions, bosons solve equation of motion

3.1 Vanishing energy

It is known that the hamiltonian of the spatially closed universe vanishes since in absence of boundaries it is given by a diffeomorphism constraint [29]. The basic argument goes as follows. In 4d space when the ansatz for the metric is taken in the form $ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j$ one finds that the hamiltonian of constraint is $H \equiv \partial S/\partial N = 0$. Here H has the contribution both from the gravity and from matter. Still one has to keep in mind that the definition of the energy of the closed universe is rather subtle.

In our new supersymmetric theory we also face the problem of how to define the energy, in general. In our case the space is not an asymptotically flat or an anti-de Sitter space. Moreover, since our space is singular, we can not easily apply the Nestor-Israel-Witten construction, which for asymptotically flat or anti-de Sitter spaces would predict a non-negative energy.

Therefore, we would perform here only a partial analysis of the energy issue for supersymmetric theories in singular spaces, which has a clear conceptual basis. We hope, however, that a more general treatment of this problem is possible.

Here we limit ourselves to configurations which depend only on x^5. For such configurations the natural definition of the energy functional was suggested and studied in [5, 7, 9]. We are using the warped metric in the form

$$ds^2 = a^2(x^5) dx^{\underline{\mu}} dx^{\underline{\nu}} \eta_{\underline{\mu\nu}} + (dx^5)^2 \,. \tag{3.1}$$

Starting with the new lagrangian (2.26), we may present the energy functional for static x^5-dependent bosonic configurations as follows

$$\begin{aligned} E(x^5) = & -6a^2 a'^2 + \frac{1}{2} a^4 (\varphi^{x\prime})^2 + a^4 V - \frac{1}{4!} \varepsilon^{\mu\nu\rho\sigma 5} A_{\mu\nu\rho\sigma} G' + \\ & + 2g \left(\delta(x^5) - \delta(x^5 - \tilde{x}^5)\right) \left(3a^4 \alpha W + \frac{1}{4!} \varepsilon^{\underline{\mu\nu\rho\sigma}} A_{\underline{\mu\nu\rho\sigma}}\right) . \end{aligned} \tag{3.2}$$

Here $'$ means $\partial/\partial x^5$. This expression in turn can be given in the BPS-type form closely related to [5, 7, 9] but still different due to (i) the presence of the 4-form and the supersymmetry singlet off shell, (ii) the presence of $\alpha = \pm$, which comes from the choice of the $\mathbb{Z}_2$ action on the fermions and is introduced in (2.24).

$$E(x^5)_{BPS} = \frac{1}{2}a^4\left\{[\varphi^{x\prime} - 3\alpha G W^{,x}]^2 - 12\left[\frac{a'}{a} + \alpha G W\right]^2\right\} + 3\alpha[a^4 G W]' + \\ + \left[2g\left(\delta(x^5) - \delta(x^5 - \tilde{x}^5)\right) - G'\right]\left(3a^4\alpha W + \frac{1}{4!}\varepsilon^{\underline{\mu\nu\rho\sigma}}A_{\underline{\mu\nu\rho\sigma}}\right). \quad (3.3)$$

The tension of the first brane, $T^1_{x^5=0}$, is equal to $6g\alpha W(x^5 = 0)$. The tension of the second brane, $T^2_{x^5=\tilde{x}^5}$, is given by $-6g\alpha W(x^5 = \tilde{x}^5)$. Even if any of them is negative, this causes no problem since we have a compensating contribution to the energy on each brane. In presence of the supersymmetry singlet field, there is an additional contribution to the energy at each brane due to the gradient of the supersymmetry singlet field G. The term with G' cancels the tension contributions at each brane separately since $G' = 2g\left(\delta(x^5) - \delta(x^5 - \tilde{x}^5)\right)$ due to the field equations for the 4-form. With account of the A and G equations and no boundary condition, which allows to ignore $+3\alpha[a^4 G W]'$, the energy functional takes the form

$$E(x^5)_{BPS} = \frac{1}{2}a^4\left\{[\varphi^{x\prime} - 3\alpha G W^{,x}]^2 - 12\left[\frac{a'}{a} + \alpha G W\right]^2\right\}. \quad (3.4)$$

Note that the *energy functional is still not positive definite*: one perfect square with the kinetic energy of the normal scalar φ is positive, however that with the kinetic energy of the conformal factor of the metric is negative, as it should be. One might have a concern about some configurations where the negative contribution will dominate over the positive one, which will lead to the instability of the theory. However, using the equations of motion for g_{00} and $A_{\mu\nu\rho\sigma}$ (not the BPS equations), one finds that the energy vanishes for any x^5-dependent solution of the equations of motion. This can also be reinterpreted as derived using the canonical formulation of the gravitational theory where one starts from

$$\int_M -\frac{1}{2}\sqrt{g}R + \int_{\partial M} \mathcal{K} = \int \tilde{\mathcal{L}}\,, \quad (3.5)$$

where $\mathcal{K}$ is such that the resulting action has no second derivatives on the metric. The action of matter fields can be added and as in [29] one finds that an on-shell energy E, at least for time-independent configurations, vanishes

$$\oint E = 0 \quad (3.6)$$

for any solution of the equations of motion.

One very important property of the vanishing of the energy of the closed universe is that it vanishes locally and not due to the compensation of the total energy, positive and negative, in different parts of the universe. As we explained this happens also here: the cancellation of energy on each brane takes case separately: the tension term is cancelled by the energy of the supersymmetry singlet field $G(x)$.

3.2 BPS construction in singular spaces

The squared terms in (3.3) suggest the BPS conditions:

$$g_{xy}\,(\varphi^y)' = +3\alpha G(x^5) W_{,x}\,, \qquad \frac{a'}{a} = -\alpha G(x^5) W\,, \tag{3.7}$$

where we can use the on shell value of $G(x^5)$, which is equal to $g\varepsilon(x^5)$, so that we obtain

$$g_{xy}\,(\varphi^y)' = +3\alpha g\varepsilon(x^5) W_{,x}\,, \qquad \frac{a'}{a} = -\alpha g\varepsilon(x^5) W\,. \tag{3.8}$$

From now on we will make the choice $\alpha = 1$, i.e. pick up a particular property of fermions under parity. Physics depends on the sign of αg. Therefore, the change in the sign of α can always be compensated by a change of the sign of g.

One can verify that the jump conditions on the branes,[7] derived starting with the second order differential equations, are satisfied automatically due to the new supersymmetry (2.29), the on-shell condition (2.18) and the presence of the 5-form flux, changing the sign when passing through the wall.

Let us also consider the Killing spinors. In the background with only non-vanishing scalars and the warped metric (3.1), which has as only non-zero components of the spin connection $\omega_{\underline{\mu}}^{m5} = a'\delta_{\underline{\mu}}^m$, the transformations of the spinors are

$$\begin{aligned}
\delta(\epsilon)\lambda_i^x &= -\mathrm{i}\frac{1}{2}\gamma_5 \varphi^{x\prime}\epsilon_i - \frac{3}{2} G W^x Q_{ij}\epsilon^j\,,\\
\delta(\epsilon)\psi_{\underline{\mu} i} &= \partial_{\underline{\mu}}\epsilon_i + \frac{1}{2}\delta_{\underline{\mu}}^m \gamma_m \left(a'\gamma_5\epsilon_i + \mathrm{i} a G W Q_{ij}\epsilon^j\right),\\
\delta(\epsilon)\psi_{5i} &= \epsilon_i' + \frac{1}{2}\mathrm{i} G W \gamma_5 Q_{ij}\epsilon^j\,.
\end{aligned} \tag{3.9}$$

To solve these, we split

$$\begin{aligned}
\epsilon_i &= \epsilon_i^+ + \epsilon_i^-\,,\\
\epsilon_i^\pm &= \frac{1}{2}\left(\epsilon_i \pm \mathrm{i}\gamma_5 Q_{ij}\epsilon^j\right) = \pm \mathrm{i}\gamma_5 Q_{ij}\epsilon^{\pm j}\,.
\end{aligned} \tag{3.10}$$

[7] It was observed in [30] that in $N=8$ gauged supergravity the jump conditions on the branes may be satisfied if the tensions are related to the superpotential. However, putting the step functions in supersymmetry transformations by hand may in general cause problems with higher order corrections. Our work makes it plausible that $N=8$ gauged supergravity with addition of the flux and supersymmetry singlet may be constructed with the complete and consistent supersymmetry. This will generate the step functions in supersymmetry rules in presence of branes.

Using the conditions (3.7), the last equation of (3.9) gives the dependence of the supersymmetries on x_5. We obtain

$$\epsilon_i^{\pm} = a^{\pm 1/2}\epsilon_i^{\pm}(x^{\underline{\mu}})\,. \tag{3.11}$$

The second equation gives

$$\partial_{\underline{\mu}}\epsilon_i^{+} + \partial_{\underline{\mu}}\epsilon_i^{-} + \delta_{\underline{\mu}}^{m}\gamma_m a'\gamma_5\epsilon_i^{-} = 0\,. \tag{3.12}$$

The solutions are thus

$$\epsilon_i = a^{1/2}\epsilon_i^{+(0)} + a^{-1/2}\left(1 - \frac{a'}{a}x^{\underline{\mu}}\gamma_{\underline{\mu}}\gamma_5\right)\epsilon_i^{-(0)}\,, \tag{3.13}$$

where $\epsilon_i^{\pm(0)}$ are constant spinors with each only 4 real components due to the projection (3.10). Remains the first Killing equation, which implies

$$\varphi^{x\prime}\epsilon_i^{-(0)} = 0\,. \tag{3.14}$$

There are thus two possibilities to solve the Killing equations.

- Maximal unbroken supersymmetry in the bulk ($N = 2$).

  Here we require that the scalars are strictly constant, $\varphi^{x\prime} = 0$, and the superpotential is independent of the scalars, $\partial W/\partial\varphi^x = 0$, at the solution. No constraints on Killing spinors arise from the gaugino.

  From the gravitino transformation, we have shown in (3.13) that in the warped geometry a result similar to [31] takes place, with doubling of supersymmetries: the constant spinors $\epsilon_i^{\pm(0)}$ give together 8 unbroken supersymmetries.

- 1/2 of the maximal unbroken supersymmetry in the bulk ($N = 1$).

  When the scalars are not constant, then there is the extra condition (3.14), leaving just one projected supersymmetry with 4 real components,

$$\epsilon_i - \mathrm{i}\gamma_5 Q_{ij}\epsilon^j = 0\,, \qquad \epsilon_i = a^{1/2}\epsilon_i^{(0)}\,, \tag{3.15}$$

  where $\epsilon_i^{(0)}$ is constant. Note that this projection of the supersymmetries is on the brane consistent with (2.25). Thus, we remain in the bulk as well as on the brane with 1/2 of the original supersymmetries. Vice versa, imposing the projection (3.15) one derives from the vanishing of (3.9) that the conditions (3.7) should be satisfied.

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3.2.1 Fixed scalars, doubling of supersymmetries and an alternative to compactification world brane

The field equation for the 4-form and G-field Killing equations are solved if (3.7) are solved. In this section we look for the very particular solutions of these equations with maximal unbroken supersymmetry that have everywhere constant scalars.[8]

The 'fixed scalar domain wall solution' is given by

$$(\varphi^y)' = 0\,, \qquad \left(\frac{\partial W}{\partial \varphi^x}\right)_{\rm crit} = 0\,, \qquad \frac{a'}{a} = -g\varepsilon(x^5) W_{\rm crit}\,. \tag{3.16}$$

The solution is given by the supersymmetric attractor equation [34, 33] in the form

$$C_{IJK}\bar h^J \bar h^K = q_I\,, \tag{3.17}$$

where

$$\bar h^K \equiv \sqrt{W_{\rm crit}} h^K\,, \tag{3.18}$$

and we have used charges normalized as

$$q_I \equiv \sqrt{\frac{2}{3}} V_I\,, \qquad \text{such that} \qquad W = h^I q_I\,, \qquad W_{,x} = -\sqrt{\frac{2}{3}} h^I_x q_I\,. \tag{3.19}$$

Consistency implies

$$W_{\rm crit}(C_{IJK}, q_I) = h^I_{\rm crit} q_I = (\bar h^I q_I)^{2/3} = (C_{IJK}\bar h^I \bar h^J \bar h^K)^{2/3}\,. \tag{3.20}$$

In some cases the explicit solution of the attractor equation is known in the form

$$\bar h^I(C_{IJK}, q_I) \tag{3.21}$$

(see e.g. [35, 33] where many examples are given). The metric is

$$ds^2 = e^{-2gW_{\rm crit}|x^5|} dx^{\underline{\mu}} dx^{\underline{\nu}} \eta_{\underline{\mu\nu}} + (dx^5)^2\,. \tag{3.22}$$

If we choose $gW_{\rm crit}$ to be positive (which means that at $x^5 = 0$ we have a positive tension brane), the metric is that introduced in [2], where two branes are present at some finite distance from each other. We will refer to this scenario as RSI. The second brane has a negative tension and can be sent to infinity, in principle, which leads to an alternative to compactification, discussed in [3]. We will refer to this as RSII. Whether this limit is totally consistent is an independent issue, however the warp factor in the metric can be chosen to exponentially decrease away from the positive tension brane. This is not in a contradiction with the no-go theorem [9, 10]. For constant scalars at the critical point, the metric behaves differently from the

[8]These solutions remind the so called double-extreme black holes [32, 33], which have fixed scalars in their solutions [34].

case when scalars are not constant but approaching the critical point. This can also be explained by the *doubling of unbroken supersymmetries in the bulk* for these solutions. Indeed, the gaugino transformations are vanishing without any constraint on the Killing spinors and the gravitino transformations also have an 8-dimensional zero mode as shown in (3.13). Note that the curvature is everywhere constant, except on the branes where the metric has a cusp. For example, the scalar curvature for this solution, is equal to

$$\frac{1}{4}R = 5g^2 W_{\rm crit}^2 - g(\delta(x^5) - \delta(x^5 - \tilde{x}^5))W_{\rm crit}\,. \tag{3.23}$$

The simplest example of fixed scalar domain wall (related to double-extreme STU black holes) comes out from the M5-brane compactified on T^6 so that $C_{IJK}h^I h^J h^K = STU = 1$ and $W_{\rm crit} = 3(q_S q_T q_U)^{1/3}$ (see [33, section 4.3]). Here q_S, q_T, q_U are FI terms.

3.2.2 BPS equations in very special geometry (with vector multiplets)

In the context of our present work in the more general case with vector multiplets present and non-constant scalars, we will first change coordinates, write down the energy functional in the new coordinate system and proceed by solving the BPS conditions following from the energy functional. We take

$$ds^2 = a^2(y)dx^{\underline{\mu}}dx^{\underline{\nu}}\eta_{\underline{\mu\nu}} + a^{-4}(y)dy^2\,, \tag{3.24}$$

so that $\partial/\partial x^5 = a^2\partial/\partial y$ and we will use " $\dot{}$ " for $\partial/\partial y$. The BPS-type energy functional for static y-dependent bosonic configurations following from the new lagrangian (2.26) is

$$E(y) = \frac{1}{2}a^2\left\{[a^2\dot{\varphi}^x - 3GW^{,x}]^2 - 12[a\dot{a} + GW]^2\right\} + 3\frac{d}{dy}[a^4 GW] + \\ + \left[2g\left(\delta(y^5) - \delta(y^5 - \tilde{y}^5)\right) - \dot{G}\right]\left(3a^4\alpha W + \frac{1}{4!}\varepsilon^{\underline{\mu\nu\rho\sigma}}A_{\underline{\mu\nu\rho\sigma}}\right).$$

The stabilization equations for the energy are the BPS conditions:

$$a^2 g_{xy}\dot{\varphi}^y = +3G(y)W_{,x}\,, \qquad a\dot{a} = -G(y)W\,, \tag{3.25}$$

where on shell for the 4-form field

$$G(y) = g\varepsilon(y)\,, \qquad \dot{G} = 2g\left(\delta(y) - \delta(y - \tilde{y})\right). \tag{3.26}$$

The solution of these closely related equations is known both in the context of black holes [36] and domain walls [37] in smooth supergravities. The difference is that we have now derived the BPS equations from supersymmetry with the step functions, and the signs are correlated with the choice of the parity assignments for fermions. This takes care of the jump conditions on the branes where our solutions have kinks.

The equations (3.25) can be combined to one $(n+1)$-component equation with a free index I in very special geometry. The equations (B.3) imply for any derivative

$$g_{xy}\dot{\varphi}^y = -\sqrt{\frac{3}{2}}h_{Ix}\dot{h}^I = \sqrt{\frac{3}{2}}h^I_x\dot{h}_I\,. \tag{3.27}$$

The BPS equations are then

$$\begin{aligned} h^I_x\left(a^2\dot{h}_I + 2q_I G(y)\right) &= 0\,, & 2a\dot{a} + 2h^I q_I G(y) &= 0\,, \\ h^I_x\left(\frac{d}{dy}(a^2 h_I) + 2q_I G(y)\right) &= 0\,, & h^I\left(\frac{d}{dy}(a^2 h_I) + 2q_I G(y)\right) &= 0\,. \end{aligned} \tag{3.28}$$

These $n+1$ equations are equivalent to

$$\frac{d}{dy}(a^2 h_I) + 2q_I G(y) = 0\,. \tag{3.29}$$

We can rewrite it using the expression for the dual coordinate $h_I = C_{IJK}h^J h^K$

$$\frac{d}{dy}(C_{IJK}\tilde{h}^J\tilde{h}^K) = -2G(y)q_I \qquad \text{where} \qquad \tilde{h}^I \equiv a(y)h^I\,. \tag{3.30}$$

3.2.3 Supersymmetric Domain Walls with non-constant scalars

Using the on-shell expression for $G(y)$ we may solve this and get that

$$C_{IJK}\tilde{h}^J\tilde{h}^K = H_I(y)\,, \tag{3.31}$$

where H_I is a harmonic function,

$$H_I = c_I - 2gq_I|y|\,, \tag{3.32}$$

satisfying the equation

$$\frac{d}{dy}\frac{d}{dy}H_I = -4gq_I[\delta(y) - \delta(y-\tilde{y})]\,. \tag{3.33}$$

This is an explicit answer for a given C_{IJK} and H_I under the condition that we know how to solve the algebraic attractor equation (3.17). If we know the solution of the algebraic attractor equation (3.17) in the form (3.21), we simply replace the q_I in this expression by the harmonic functions $H_I(y)$.

$$\tilde{h}^I(y) = \tilde{h}^I(C_{IJK}, H_I(y))\,, \tag{3.34}$$

and we can find the scalars and the metric. Contracting the attractor equation with $\tilde{h}^I$ we find that

$$C_{IJK}\tilde{h}^I\tilde{h}^J\tilde{h}^K = a^3(y)C_{IJK}h^I h^J h^K = a^3(y) \qquad \text{or} \qquad a^2(y) = h^I H_I\,. \tag{3.35}$$

Our solution for the domain wall metric is therefore given by (3.24) with $a(y)$ given by this expression.

The choice of coordinates for the scalars φ^x has been left unspecified so far. One possible choice is that similar to the 'special coordinates' in $d = 4$, $N = 2$:

$$\phi^I = \frac{\tilde{h}^I(y)}{\tilde{h}^0(y)} = \{1, \varphi^x(y)\}\,. \tag{3.36}$$

The general class of domain wall solutions as described in the previous section has many particular realizations, depending on the choice of the intersection numbers C_{IJK} as well as on the choice of the initial conditions at $y = 0$ defined by the first terms in the harmonic functions, c_I. Only if at least some of the c_I are not vanishing, would the scalars depend on y. Otherwise all ratios of harmonic functions will be y-independent and therefore all scalars will be constants and the solutions will be those from the previous section.

The properties of the solutions with non-constant scalars depend on the choice of the model, i.e. the choice of the intersection numbers C_{IJK}, and also on various choices of the signs between the first and the second term in the harmonic functions. The options are:

- All c_I have the opposite sign to all gq_I, i.e. none of the harmonic functions vanishes at some value of y, $|H_I| = |c_I| + |2gq_I||y| > 0$. In such case one can directly use most of the solutions of the stabilization equations from [33] and present the relevant domain walls. Particularly interesting ones require the second wall to cut off the part of the 5D space where the relevant CY cycles collapse and/or the metric of the moduli space vanishes.

- At least some of the c_I have the same sign as gq_I, e.g. for positive c_1 and positive gq_1, $H_1 = c_1 - 2gq_1|y|$ vanishes at $|y|_{\rm sing} = c_1/2gq_1$. Such cases definitely require the second brane at $|\tilde{y}|$ to be placed before $|y|_{\rm sing}$ since at $|y|_{\rm sing}$ the metric of the moduli space and the space-time metric may vanish.

3.3 Examples

Here we briefly present a couple of examples. It would be interesting to undertake a more careful study of these solutions.

3.3.1 STU Wall

We consider again $C_{IJK}h^I h^J h^K = STU = 1$ and the harmonic functions are given by

$$H_S = c_S - 2gq_S|y|\,, \qquad H_T = c_T - 2gq_T|y|\,, \qquad H_U = c_U - 2gq_U|y|\,. \tag{3.37}$$

If all c are positive and all gq are negative, all harmonic functions are strictly positive. The metric is

$$ds^2 = 3(H_S H_T H_U)^{1/3}(dx)^2 + 3^{-2}(H_S H_T H_U)^{-2/3}dy^2\,, \tag{3.38}$$

and the moduli are

$$S = \left(\frac{H_T H_U}{H_S^2}\right)^{1/3}, \qquad T = \left(\frac{H_S H_U}{H_T^2}\right)^{1/3}, \qquad U = \left(\frac{H_T H_S}{H_U^2}\right)^{1/3}. \tag{3.39}$$

We use the normalization $3(c_S c_T c_U)^{1/3} = 1$ so that near $|y| = 0$ the domain wall metric (3.24) tends to a flat Minkowski metric.

$$ds^2_{y\to 0} \Rightarrow dx^{\underline{\mu}} dx^{\underline{\nu}} \eta_{\underline{\mu\nu}} + dy^2 \,. \tag{3.40}$$

If near the second wall at $|y| = |\tilde{y}|$ the values of the second terms in the harmonic functions are much larger than the first ones, the metric near this brane approaches the boundary of the adS space with

$$ds^2 \Rightarrow \rho^2 dx^{\underline{\mu}} dx^{\underline{\nu}} \eta_{\underline{\mu\nu}} + R^2_{\mathrm{AdS}} \left(\frac{d\rho}{\rho}\right)^2, \tag{3.41}$$

where $y = \frac{1}{2} R_{\mathrm{AdS}} \rho^2$ and $R^{-2}_{\mathrm{AdS}} = 9 g^2 (q_S q_T q_U)^{2/3}$.

If in any of the harmonic functions the sign between the terms is opposite, and the harmonic function vanishes at $|y|_{\mathrm{sing}}$, the second brane has to be at the position $|\tilde{y}| < |y|_{\mathrm{sing}}$.

3.3.2 Calabi-Yau wall (base $\mathbf{P}_2$ vacuum)

Here we take an example of a large complex structure limit of a particular Calabi-Yau threefold [38] which defines a cubic form for 5D supergravity.

$$\mathcal{V}(h^1, h^2) = 9(h^1)^3 + 9(h^1)^2 h^2 + 3h^1 (h^2)^2 = 1\,. \tag{3.42}$$

The attractor equation for this theory was solved in [33]. We will use this solution to present one for the CY domain wall. We perform a coordinate redefinition to variables U and T, related to the basic cycles h^1 and h^2 by $U = 6^{1/3} h^1$, $T = 6^{1/3}(h^2 + \frac{3}{2} h^1)$ and

$$\mathcal{V}(U, T) = \frac{3}{8} U^3 + \frac{1}{2} U T^2 = 1\,. \tag{3.43}$$

The relevant harmonic functions are given by

$$H_U = c_U - 2 g q_U |y|\,, \qquad H_T = c_T - 2 g q_T |y|\,, \tag{3.44}$$

and we choose $H_U > 0$ and $H_T > 0$. The domain wall metric takes the form (3.24), where

$$a^2(|y|) = \left[H_U \frac{2}{\sqrt{3}} \sqrt{H_U - \sqrt{H_U^2 - \frac{9}{4} H_T^2}} + H_T \sqrt{3} \sqrt{H_U + \sqrt{H_U^2 - \frac{9}{4} H_T^2}} \right]^{2/3}. \tag{3.45}$$

The moduli are given by

$$U(|y|) = \frac{2}{\sqrt{3}a(|y|)}\sqrt{H_U - \sqrt{H_U^2 - \frac{9}{4}H_T^2}}\,, \qquad T(|y|) = \frac{\sqrt{3}}{a(|y|)}\sqrt{H_U + \sqrt{H_U^2 - \frac{9}{4}H_T^2}}\,. \tag{3.46}$$

The range of the variables is determined by the positivity of the basic cycles h^1 and h^2, which translates for our solution into

$$2H_U - 3H_T > 0\,, \qquad H_U > 0\,, \qquad H_T > 0\,. \tag{3.47}$$

To satisfy this condition at $|y| = 0$ we have to require that $c_U > 0$, $c_T > 0$ and $2c_U - 3c_T > 0$. Note that if H_T would vanish, which may happen for $gq_T > 0$ and $c_T > 0$ at $|y|_{\rm sing} = c_T/2gq_T$, this would be the point where the cycle h^1 would collapse. Indeed at $H_T = 0$, the ratio $U/T = \frac{h^1}{h^2+(3/2)h^1} = 0$. At this point also $a(|y|_{\rm sing}) = 0$ and the spacetime metric is singular. To avoid such a singularity we may require that the second brane is at

$$|\tilde{y}| < |y|_{\rm sing} = \frac{c_T}{2gq_T}\,. \tag{3.48}$$

Even if none of the harmonic functions H_U and H_T vanishes, i.e. gq_T and gq_U are both negative, we still have an inequality

$$2c_U - 3c_T - |y|2g(2q_U - 3q_T) > 0\,. \tag{3.49}$$

If the condition $g(2q_U - 3q_T) > 0$ is satisfied, then there is also an upper limit for the distance between the branes:

$$|\tilde{y}| < \frac{2c_U - 3c_T}{2g(2q_U - 3q_T)}\,. \tag{3.50}$$

At this point $h^2 = 6^{-1/3}(T - \frac{3}{2}U) = 0$, i.e. the second cycle is collapsing. Thus to prevent this from happening, we have to put the second wall before this value of $|y|$.

As we already mentioned, a more careful and detailed analysis of domain walls of section 3.2.3 in application to other CY spaces, as it was done for CY black holes, may lead to more interesting and diverse configurations.

4. Discussion: 10D supersymmetry and 8-branes on the orbifold

The massive 10D supergravity theory of Romans [39] describing the 8-brane is expected to have the same basic features as 5D supergravity describing the 3-brane. Thus we expect using [40, 24] and the results of the present paper that the 10D supersymmetry on $S^1/\mathbb{Z}_2$ can be realized as follows:

$$S_{\rm new} = S_{\rm bulk} + S_{\rm brane}\,, \qquad S_{\rm bulk} = S_{\rm Romans}(m \to M(x)) + S_A = S_{BGPT} + \text{fermions}\,. \tag{4.1}$$

Here $S_{\rm Romans}(m \to M(x))$ is the Romans action [39] where the constant mass m is replaced by a field $M(x)$ and the 9-form field is added in $S_A = \int \epsilon^{(10)} dA_9 M$ so that the total bosonic bulk action is that given by Bergshoeff, Green, Papadopoulos and Townsend in [24] (we have not given the fermion terms which can be taken from the Romans action in [39]). Note that a slightly different version of the bosonic bulk plus brane action, including a kinetic term for the 9-form potential and a brane kinetic term for the BI vector fields, has been given in [40].[9]

The brane action has to be added to the bulk action so that the on-shell value of the M-field is piecewise constant. In particular, for an $S^1/\mathbb{Z}_2$ orbifold, the value of M should change sign across the wall. The parity assignments of the fields follow straightforwardly by reducing the $D = 11$ parity assignments under $y \to -y$, where y refers to the direction between the "end of the world" branes of Hořava and Witten, over a direction different from y, i.e. a direction inside the branes.

Without the brane action, the on-shell field M is the same constant everywhere and this prevents us from getting the harmonic functions depending on moduli $|y|$. The brane actions for the 8-brane are expected to contain the following terms

$$\mathcal{L}_{\rm brane} \sim -2m \; (\delta(y) - \delta(y - \tilde{y})) \left(e_{(9)} e^{5\sigma/4} + \frac{1}{9!} \varepsilon^{\underline{\mu}_1 \cdots \underline{\mu}_9} A_{\underline{\mu}_1 \cdots \underline{\mu}_9} \right), \tag{4.2}$$

where σ is the Romans scalar which is related to the standard dilaton ϕ via $\sigma = -4/5\phi$ such that the brane tension is proportional to $1/g_s = e^{-\langle\phi\rangle}$. The coordinate y refers to the direction connecting the two branes. The modified supersymmetry transformations in the presence of the supersymmetry singlet field M and of the 9-form are expected to exist and the bulk and the brane actions will be supersymmetry invariant. Note that the brane action only contains bosonic terms. It is supersymmetric due to the orbifold condition $\psi_{\underline{\mu}}(y) = \gamma_y \psi_{\underline{\mu}}(-y)$.

The new physics due to the presence of the brane action will show up via the M-field equation. The bosonic part of this equation following from the bulk action is given in [24] and reads

$$\hat{F} \equiv \epsilon \hat{F}_{10} = M(x) e K(B)\,. \tag{4.3}$$

Here K is a polynomial in terms of the massive 2-form field B and $\hat{F}_{10}$ is the covariant curvature of A_9. With account of the brane action (4.2) added according to our rule of consistent supersymmetry in the singular spaces, the on-shell value of the M-field will be proportional to the step function:

$$\hat{F} = m\varepsilon(y) e K(B)\,. \tag{4.4}$$

Clearly the flux is changing sign when passing through the brane, according to the field equations of the theory.

[9] We obtain a kinetic term for the 9-form potential if we eliminate M as auxiliary field, rather than using the field equation for the 9-form.

One would hope that the improved supersymmetric formulation of the 8-brane of string theory in the framework of 10D supergravity in a singular space will shed some light on the M-theory and its 9-brane. Since 11D supergravity does not admit a cosmological constant, one may in this way uplift from 10D to 11D some important information on extended objects of string theory, which is difficult to understand from the 11D massless supergravity point of view.

A related general problem to which the construction of this paper may be useful is to clarify the appearance of chiral fermions on the brane in the context of supersymmetric theory with extended supersymmetry in the bulk.

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A. Notations

The metric is $(-++++)$, and $[ab]$ denotes antisymmetrization with total weight 1, thus $\frac{1}{2}(ab-ba)$. We use the following indices

$$\begin{array}{lll}
\mu & 0,\ldots,3,5 & \text{local spacetime} \\
\underline{\mu} & 0,\ldots,3 & \text{4-d local spacetime} \\
a & 0,\ldots,3,5 & \text{tangent spacetime} \\
m & 0,\ldots,3 & \text{tangent spacetime in 4 dimensions} \\
i & 1,2 & \mathrm{SU}(2) \\
I & 0,\ldots,n & \text{vectors} \\
x & 1,\ldots,n & \text{scalars in vector multiplets.}
\end{array} \tag{A.1}$$

For the curvatures and connections, we use the conventions:

$$\begin{aligned}
\omega_\mu^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c\,, \\
R_{\mu\nu}{}^{ab} &= 2\partial_{[\mu}\omega_{\nu]}^{ab} + 2\omega_{[\mu}^{ac}\omega_{\nu]c}{}^b\,, \\
R_{\mu\nu} &= R_{\mu\rho}{}^{ba}e_b{}^\rho e_{\nu a}\,, \qquad R = g^{\mu\nu}R_{\mu\nu}\,, \\
R^\mu{}_{\nu\rho\sigma} &= R_{\rho\sigma}{}^{ab}e_a^\mu e_{\nu b} = 2\partial_{[\rho}\Gamma^\mu_{\sigma]\nu} + 2\Gamma^\mu_{\tau[\rho}\Gamma^\tau_{\sigma]\nu}\,.
\end{aligned} \tag{A.2}$$

For the metric (3.1), the only non-zero components of the spin connection are

$$\omega_{\underline{\mu}}^{m5} = a'\delta_{\underline{\mu}}^m\,. \tag{A.3}$$

Here $'$ means $\partial/\partial x^5$. The non-zero curvature components, Ricci tensor and scalar curvature are

$$\begin{aligned} R_{5\underline{\mu}}{}^{m5} &= a''\delta_{\underline{\mu}}^{m}\,, & R_{\underline{\mu\nu}}{}^{mn} &= -2a'^2\delta_{\underline{\mu}}^{[m}\delta_{\underline{\nu}}^{n]}\,,\\ R_{55} &= 4a^{-1}a''\,, & R_{\underline{\mu\nu}} &= (aa''+3a'^2)\eta_{\underline{\mu\nu}}\,,\\ R &= 8a^{-1}a''+12a^{-2}a'^2\,, & eR &= -12a^2a'^2+8(a'a^3)'\,. \end{aligned} \tag{A.4}$$

The Levi-Civita tensor is real, and

$$\varepsilon_{abcde}\varepsilon^{abcde} = -5!\,,\qquad \varepsilon^{\mu\nu\rho\sigma\tau} = ee_a^{\mu}e_b^{\nu}\cdots e_e^{\tau}\varepsilon^{abcde}\,. \tag{A.5}$$

The gamma matrices are related by

$$\gamma^{abcde} = \mathrm{i}\varepsilon^{abcde}\,. \tag{A.6}$$

SU(2) indices are raised and lowered with ε_{ij}, where $\varepsilon_{12} = \varepsilon^{12} = 1$, in NW-SE convention:

$$X^i = \varepsilon^{ij}X_j\,,\qquad X_i = X^j\varepsilon_{ji}\,. \tag{A.7}$$

Spinor indices are omitted. The charge conjugation $\mathcal{C}$ and $\mathcal{C}\gamma_a$ are antisymmetric. $\mathcal{C}$ is unitary and γ_a is hermitean apart from the timelike one, that is antihermitean. The bar is the Majorana bar:

$$\bar{\lambda}^i = (\lambda^i)^T\mathcal{C}\,. \tag{A.8}$$

Define the charge conjugation operation on spinors as

$$(\lambda^i)^C \equiv \alpha^{-1}B^{-1}\varepsilon^{ij}(\lambda^j)^*\,,\qquad \bar{\lambda}^{iC} \equiv \overline{(\lambda^i)^C} = \alpha^{-1}\left(\bar{\lambda}^k\right)^* B\varepsilon^{ki}\,, \tag{A.9}$$

where $B = \mathcal{C}\gamma_0$, and α is arbitrary ± 1 when you use the convention that complex conjugation does not interchange the order of spinors, or $\pm\mathrm{i}$ when complex conjugation does interchange the order of spinors. Symplectic Majorana spinors satisfy $\lambda = \lambda^C$. Charge conjugation acts on gamma matrices as $(\gamma_a)^C = -\gamma_a$, does not change the order of matrices, and works on matrices in SU(2) space as $M^C = \sigma_2 M^*\sigma_2$. Complex conjugation can then be replaced by charge conjugation, if for every bispinor one inserts a factor -1. Then e.g. the expression

$$\bar{\lambda}^i\gamma^{\mu}\partial_{\mu}\lambda_i \tag{A.10}$$

is real for symplectic Majorana spinors. For more details, see e.g. [41].

B. Supergravity in 5 dimensions with vector multiplets

We present here 5-dimensional supergravity with vector multiplets, which we will use in the main text. To put it in perspective, we repeat that the pure supergravity was constructed in [42, 43]. The coupling with vector multiplets was obtained in [11], and with gauging of the vectors in [12]. This was extended to tensor multiplets in [13], and an action coupled to an arbitrary quaternionic manifold is obtained in [14].

We consider the coupling of abelian vector multiplets to supergravity. The fields are the fünfbein, the gravitino, the vectors, scalars and gauginos:

$$e_\mu^a\,,\ \psi_\mu^i\,,\ A_\mu^I\,,\ \varphi^x\,,\ \lambda^{ix}\,. \tag{B.1}$$

The vector multiplets couplings are determined by the symmetric real constant tensor C_{IJK}. The scalars appear in the functions[10] $h^I(\varphi)$, satisfying

$$h^I(\varphi)h^J(\varphi)h^K(\varphi)C_{IJK}=1\,, \tag{B.2}$$

and define the quantities

$$\begin{aligned}
h_I(\varphi) &\equiv C_{IJK}h^J(\varphi)h^K(\varphi)=a_{IJ}h^J\,, && a_{IJ}\equiv -2C_{IJK}h^K+3h_Ih_J\,,\\
h_x^I &\equiv -\sqrt{\frac{3}{2}}h^I_{,x}(\varphi)\,, && h_{Ix}\equiv a_{IJ}h_x^J=\sqrt{\frac{3}{2}}h_{I,x}(\varphi)\,,\\
g_{xy} &\equiv h_x^Ih_y^Ja_{IJ}=-2h_x^Ih_y^JC_{IJK}h^K\,, && T_{xyz}\equiv C_{IJK}h_x^Ih_y^Jh_z^K\,,
\end{aligned} \tag{B.3}$$

with $_{,x}$ an ordinary derivative with respect to φ^x. The I-type indices are lowered or raised with a_{IJ} or its inverse, which we assume to exist, and the same holds for the metric g_{xy} used for the indices on scalars and gauginos.

Many identities have been derived in previous papers. The most useful ones for us are

$$\begin{aligned}
&h_Ih_x^I=0\,, \qquad h_Ih_J+h_I^xh_{Jx}=a_{IJ}\,.\\
&h_{Ix;y}\equiv h_{Ix,y}-\Gamma^z_{xy}h_{Iz}=\sqrt{\frac{2}{3}}\left(h_Ig_{xy}+T_{xyz}h_I^z\right),
\end{aligned} \tag{B.4}$$

with the connection Γ^z_{xy} as usual defined such that $g_{xy;z}=0$.

We consider just one gauged U(1), whose coupling constant is called g, and whose gauge vector is a linear combination of the vectors,

$$A_\mu^{(R)}\equiv V_IA_\mu^I\,, \tag{B.5}$$

where V_I are real constants. It gauges a direction in SU(2) space determined by a matrix (with real q_1, q_2 and q_3):

$$\begin{aligned}
&Q_i{}^j=\mathrm{i}(q_1\sigma_1+q_2\sigma_2+q_3\sigma_3)\,, \qquad (q_1)^2+(q_2)^2+(q_3)^2=1\,,\\
&Q_{ij}=-\mathrm{i}q_1\sigma_3-q_2\mathbb{1}+\mathrm{i}\sigma_1q_3\,, \qquad Q^{ij}=\mathrm{i}q_1\sigma_3-q_2\mathbb{1}-\mathrm{i}\sigma_1q_3\,,\\
&Q_{ij}Q^{jk}=\delta_i^k\,, \qquad Q_i{}^jQ_j{}^k=-\delta_i^k\,.
\end{aligned} \tag{B.6}$$

[10]These functions are arbitrary up to non-degeneracy conditions that the $(n+1)\times(n+1)$ matrix (h^I,h^I_x) (see definition below) should be invertible.

The equations of [12] are obtained by choosing $q_1 = q_3 = 0$, $q_2 = 1$. They, and [14], introduce also other functions that are useful when one considers non-abelian gauging:

$$P_{Iij} = V_I Q_{ij}\,, \qquad P_{ij} = h^I V_I Q_{ij}\,, \qquad P_{ij}^x = h^{xI} V_I Q_{ij}\,. \tag{B.7}$$

We will use the quantities

$$W = \sqrt{\frac{2}{3}} h^I V_I\,, \tag{B.8}$$

with derivative

$$W_{,x} \equiv \frac{\partial}{\partial \varphi^x} W = -\frac{2}{3} h_x^I V_I\,. \tag{B.9}$$

The action is then $S_{GST} = \int \mathrm{d}^5 x\, \mathcal{L}_{GST}$, with

$$\begin{aligned}
e^{-1}\mathcal{L}_{GST} = & -\frac{1}{2}R(\omega) - \frac{1}{2}\bar\psi_\mu^i \gamma^{\mu\nu\rho}\left(\nabla_\nu(\omega)\psi_{\rho i} + g A_\nu^{(R)} Q_{ij}\psi_\rho^j\right) - \frac{1}{4}a_{IJ}F_{\mu\nu}^I F^{J\mu\nu} + \\
& + \frac{1}{6\sqrt{6}} e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda} C_{IJK} F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K - \frac{1}{2} g_{xy}\partial_\mu\varphi^x\partial^\mu\varphi^y - V(x) - \\
& - \frac{1}{2}\overline{\lambda}_x^i \gamma^\mu \left(\nabla_\mu(\omega)\lambda_i^x + \Gamma_{yz}^x(\partial_\mu\varphi^y)\lambda_i^z + g A_\mu^{(R)} Q_{ij}\lambda^{jx}\right) - \\
& - \mathrm{i}\frac{\sqrt{6}}{16}\left[\bar\psi_\mu^i\gamma^{\mu\nu\rho\sigma}\psi_{\nu i}F_{\rho\sigma}^I h_I + 2\bar\psi^{\mu i}\psi_i^\nu F_{\mu\nu}^I h_I\right] + \frac{1}{4}\overline{\lambda}^{ix}\gamma^\mu\gamma^{\nu\rho}\psi_{i\mu}F_{\nu\rho}^I h_{Ix} - \\
& - \frac{\mathrm{i}}{2}\overline{\lambda}_x^i\gamma^\mu\gamma^\nu\psi_{i\mu}\partial_\nu\varphi^x + \frac{\mathrm{i}}{4}\sqrt{\frac{2}{3}}\left(\frac{1}{4}g_{xy}h_I + T_{xyz}h_I^z\right)\overline{\lambda}^{ix}\gamma^{\mu\nu}\lambda_i^y F_{\mu\nu}^I + \\
& + g Q_{ij}\left[\mathrm{i}\overline{\lambda}^{ix}\lambda^{jy}\left(-\frac{1}{4}g_{xy}W + \sqrt{\frac{3}{2}}T_{xyz}W^{,z}\right) - \right. \\
& \qquad \left. - \frac{3}{2}\overline{\lambda}_x^i W^{,x}\gamma^\mu\psi_\mu^j + \mathrm{i}\frac{3}{4}\overline{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j W\right] + \\
& + \text{4-fermion terms.}
\end{aligned} \tag{B.10}$$

where ($W^{,x} \equiv g^{xy}W_{,y}$)

$$\begin{aligned}
\nabla_\mu &= \partial_\mu + \frac{1}{4}\omega_{\mu,ab}\gamma^{ab}\,, \qquad F_{\mu\nu}^I = 2\partial_{[\mu}A_{\nu]}^I\,, \\
V(x) &= -4g^2 V_I V_J C^{IJK} h_K = g^2\left(-6W^2 + \frac{9}{2}W^{,x}W_{,x}\right).
\end{aligned} \tag{B.11}$$

The action is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta(\epsilon)e_\mu^a &= \frac{1}{2}\bar\epsilon^i\gamma^a\psi_{\mu i}\,, \\
\delta(\epsilon)\psi_{\mu i} &= \partial_\mu\epsilon_i + \frac{1}{4}\gamma^{cd}\hat\omega_{\mu,cd}\epsilon_i + g A_\mu^{(R)} Q_{ij}\epsilon^j + \frac{\mathrm{i}}{4\sqrt{6}}\left(\gamma_{\mu\nu\rho} - 4g_{\mu\nu}\gamma_\rho\right)\epsilon_i h_I \widehat{F}^{\nu\rho\, I} - \\
& - \frac{1}{12}\gamma_{\mu\nu}\epsilon^j\overline{\lambda}_i^x\gamma^\nu\lambda_{jx} + \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon^j\overline{\lambda}_i^x\gamma^{\nu\rho}\lambda_{jx} + \frac{1}{6}\epsilon^j\overline{\lambda}_i^x\gamma_\mu\lambda_{jx} - \\
& - \frac{1}{12}\gamma^\nu\epsilon^j\overline{\lambda}_i^x\gamma_{\mu\nu}\lambda_{jx} + \mathrm{i}\frac{1}{2}g\,\gamma_\mu\epsilon^j W Q_{ij}\,,
\end{aligned}$$

$$
\begin{aligned}
\delta(\epsilon)A^I_\mu &= -\frac{1}{2}\overline{\epsilon}^i\gamma_\mu\lambda^x_i h^I_x + \frac{\mathrm{i}\sqrt{6}}{4}h^I\overline{\psi}^i_\mu\epsilon_i\,,\\
\delta(\epsilon)\varphi^x &= \frac{\mathrm{i}}{2}\overline{\epsilon}^i\lambda^x_i\,,\\
\delta(\epsilon)\lambda^x_i &= -\frac{\mathrm{i}}{2}\widehat{\not{\partial}}\varphi^x\epsilon_i + \frac{1}{4}h^x_I\gamma^{\mu\nu}\epsilon_i\widehat{F}^I_{\mu\nu} - g\frac{3}{2}\epsilon^j W^{,x}Q_{ij} -\\
&\quad -\frac{\mathrm{i}}{2}h^x_I h_{Iy,z}\lambda^y_i\overline{\epsilon}^j\lambda^z_j + \frac{\mathrm{i}}{\sqrt{6}}\epsilon^j\overline{\lambda}^y_i\lambda^z_j T_{xyz}\,,
\end{aligned}
\tag{B.12}
$$

where

$$
\begin{aligned}
\hat{\omega}_{\mu ab} &= \omega_{\mu ab} - \frac{1}{4}\left(\bar{\psi}^i_b\gamma_\mu\psi_{ai} + 2\bar{\psi}^i_\mu\gamma_{[b}\psi_{a]i}\right),\\
\widehat{F}^I_{\mu\nu} &= 2\partial_{[\mu}A^I_{\nu]} + h^I_x\,\overline{\psi}^i_{[\mu}\gamma_{\nu]}\lambda^x_i + \frac{1}{4}\mathrm{i}\sqrt{6}\,h^I\,\overline{\psi}^i_{[\mu}\psi_{\nu]i}\\
\widehat{\partial}_\mu\varphi^x &= \partial_\mu\varphi^x - \frac{\mathrm{i}}{2}\overline{\psi}^j_\mu\lambda^x_j\,.
\end{aligned}
\tag{B.13}
$$

C. Previous work on supersymmetry in singular spaces

The first idea to have two supersymmetric domain walls relating the 11-dimensional physics with the 10-dimensional one, was suggested in HW [17]. They made the connection between the bulk and the branes at the quantum level, i.e. 1-loop anomalies of the bulk action are cancelled by a classical non-invariance of the action on the brane. The non-trivial flux that introduces the step functions into the theory, is related to an anomaly. The realization of the supersymmetry of this combined bulk and brane actions was successful at the order $\kappa^{2/3}$, which was established by rather involved calculations. However, at the order $\kappa^{4/3}$ things go out of control. Some $\delta(0)$ terms occur. The hope was expressed that a proper quantum M-theory treatment will lead to quantum consistency. The important aspect of this theory was the appearance of chiral fermions on the brane.

The next step in this development was undertaken by [18], where HW theory was compactified on a CY space and the "5D supergravity" with the step functions in the supersymmetry rules, was recovered. The supersymmetry in this approach relies on the supersymmetry of HW theory and therefore would be difficult to complete.

More recently, two new approaches to realize the supersymmetric RS scenario were suggested in the framework of 5D supergravity: one in [19] and an alternative one in [20, 22]. The basic difference is that in [19] the gauge coupling constant is even across the wall and in [20, 22] it is taken to be odd as in HW theory. We have shown in section 2.1, using a standard basis for symplectic spinors, that there are indeed two possibilities to make the $\mathbb{Z}_2$ symmetry on spinors commuting with the supersymmetry: one with even gauge coupling, when the projector in parity rules anticommutes with the projector in supersymmetry rules, and the second one with the odd gauge coupling, when the projector in parity rules commutes with the projector in supersymmetry rules. So far both versions were fine.

In the case of even g [19], the bulk action is simply the action of supergravity, without step functions: therefore the standard supersymmetry rules are valid for the bulk action and there is no compelling reason to add the brane actions. The motivation for adding the brane actions comes from the properties of the singular background: if one would consider configurations with unbroken supersymmetries for domain walls, one would not solve Killing equations everywhere. This enforces some particular changes in supersymmetry rules at the singularities only. The brane actions are added to compensate the supersymmetry of the bulk action, which is not supersymmetric after the δ-functional changes in the supersymmetry rules. We have checked that if one would start not from pure supergravity without matter, as in [19], but with additional multiplets, the change of supersymmetry rules would depend on the scalars in the background and the issue of the closure of the algebra would become obscure as the scalars had to satisfy equations of motion. The conclusion of this analysis was that it will be difficult to generalize the approach of [19] in pure supergravity to more general theories with matter multiplets.

We therefore decided to continue along the line of [20, 22], where the gauge coupling g was taken to be odd. This enforced us to introduce step functions in the supersymmetry rules and as a consequence the bulk supergravity action was not supersymmetric anymore: the brane action of [22] was designed to compensate the problem of the bulk action. This was conceptually and technically correct from our perspective. The problem was how to make the total system complete. Later when the supersymmetry singlet and the 4-form potential were introduced, we were able to construct a consistent and complete supersymmetric theory where the bulk and the brane actions are supersymmetric. Remarkably, if we would not care about the conceptual issues and would only try to accomplish the complete supersymmetric theory, we would have to use the odd gauge coupling to have a consistent implementation of the 4-form field, see section 2.2.3, where this is explained.

Here we also would like to comment on a very interesting paper [21]. It does not deal with fermions, therefore it does not address directly the issues discussed in our paper. However, it brings some deep insights from the perspective of the brane actions: in particular the role of the $(D-1)$-form is stressed as required for the Wess-Zumino terms of codimension 1 branes, as well as the fact that the bulk action has to be supplemented by the term that is quadratic in the D-form flux. In our supersymmetric theory, we have not eliminated the field G as an auxiliary field by its equation of motion. If we would have done it using the action in the form of (2.27) and (2.28), we would find out that the bulk action has a term quadratic in the flux $\hat{F}$. We expect that our supersymmetric action with the brane action where the worldvolume degrees of freedom are not excited, may be further developed in the spirit of [21] and lead to a deeper understanding of the total dynamical system.

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